Decomposition of $Spin^c(4)$ Gauge Potential and Determinant Equation for Twisting $U(1)$ Potential in Seiberg-Witten Theory

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Abstract

The Seiberg-Witten equations are studied from the viewpoint of gauge potential decomposition. We find a determinant equation $\Delta A_\mu = -\lambda A_\mu$ for the twisting $U(1)$ potential A_μ of the Seiberg-Witten theory, which is in itself an eigenvalue problem of the Laplacian operator, with the eigenvalue being the vacuum expectation value of the field function, $\lambda = \|\Phi\|^2/2$. This establishes a direct relationship between the spectral theory of the Laplacian operator and the classification of the moduli space of the Dirac operator. Topological characteristic numbers of instantons in the self-dual $SU(2)_+$ sub-space are also discussed.

1 Introduction

Decomposition theories of gauge potentials have become a useful tool in physicists' mathematical arsenal [1, 2, 3, 4]: It is known in mathematics that for a principal bundle of structure group G, topological singularities of the base manifold may be characterized by a basic field function ψ distributed whereon (e.g., a vector, spinor, or metric field of a physical system). This allows one to decompose the gauge potential (connection) ω of the physical

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system in terms of ψ, and analyze inner structures and singularities of topological characteristic classes constructed with the field strength (curvature) $F = F(\omega)$. In this paper the Seiberg-Witten (SW) equations will be studied by investigating the inner structure of the $Spin^c(4)$ gauge potential.

The SW theory determines the exact low-energy effective Lagrangian of the $\mathcal{N} = 2$ supersymmetric Yang-Mills gauge theory in terms of a single prepotential $\mathcal{F}$ defined from the SW curve [5]. It resulted in a revolution in string theory in 1994, and also played an important role in topology by providing a powerful SW invariant for the classification of four-dimensional manifolds [6].

Let $\mathcal{M}$ be an oriented Riemannian 4-manifold and P a principal $Spin^c(4)$-bundle on $\mathcal{M}$, $Spin^c(4) = Spin(4) \otimes S^1/\mathbb{Z}_2$. The SW equations read

$$\mathbf{D}^{+A}\Psi = \gamma^\mu\partial_\mu\Psi - \gamma^\mu\omega_\mu\Psi - i\gamma^\mu A_\mu\Psi = 0, \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -\bar{\Psi}\gamma_{\mu\nu}\Psi, \tag{2}$$

where $\mu, a = 1,2,3,4$ denote the base manifold and Clifford algebraic indices respectively; γ_μ is obtained from the Dirac matrices γ_a via the vierbein $\gamma_\mu = e_{\mu a}\gamma_a$, $\gamma_{\mu\nu} = e_{\mu a}e_{\nu b}I_{ab}$ with $I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ being the $SO(4)$ [i.e. $Spin(4)$] generators, $\omega_\mu = \frac{1}{2}\omega_{\mu ab}I_{ab}$ is the $SO(4)$ gauge potential, $\omega_{\mu ab} = -\omega_{\mu ba}$, and A_μ is the $U(1)$ potential of the determinant line bundle L. The Dirac 4-spinor Ψ is a self-dual solution of the twisted Dirac operator

$$\mathbf{D}^{+A} : \Gamma(S^+ \otimes L) \to \Gamma(S^- \otimes L),$$

where $S^\pm \otimes L$ are the twisted self- and anti-self-dual sub-spaces respectively. Eq.(2) is the so-called monopole equation.

In Sect.2 below, we start with (1) and briefly discuss the decomposition of the $Spin^c(4)$ gauge potential $\omega_\mu + iA_\mu$ in terms of the Dirac spinor Ψ. The result is compared in Sect.3 with the Pauli 2-spinor decomposition of the $SU(2)$ potential. This leads to the discovery of the determinant equation (14) for the twisting $U(1)$ potential A_μ, which is in itself an eigenvalue problem of the Laplacian operator, with eigenvalue being the gauge-invariant vacuum expectation value of the field Ψ. This reveals a direct connection of the spectral theory of the Laplacian operator with the classification of the moduli space of the Dirac operator in the SW equations. In Sect.4, we discuss topological characteristic numbers of instantons in the self-dual $SU(2)_+$ subspace of $Spin^c(4)$, and conclude the paper by presenting a summary and discussion.

2 Decomposition of $Spin^c(4)$ Gauge Potential

Our starting point is the SW equation (1). We will treat in this section the wave function Ψ as an ordinary Dirac spinor, instead of a self-dual one. Then (1) is a general covariant Dirac equation describing massless neutrinos under an electromagnetic field. Introducing $\omega_{abc} = e_a^\mu \omega_{\mu bc}$ and $\gamma_a \partial_a = \gamma^\mu \partial_\mu$ by horizentally lifting $x_a = e_a^\mu x_\mu$, (1) becomes

$$\gamma_a \partial_a \Psi - \frac{1}{4}\omega_{abc}\gamma_a\gamma_b\gamma_c\Psi - iA_\mu\gamma^\mu\Psi = 0.$$

One notices that ω_{abc} may be written in three parts,

$$\omega_{abc} = \omega_{abc}^A + \omega_{abc}^{S_1} + \omega_{abc}^{S_2},$$

where ω_{abc}^A is fully anti-symmetric for $a\ b\ c$, $\omega_{abc}^{S_1}$ symmetric for $a\ b$, and $\omega_{abc}^{S_2}$ symmetric for $a\ c$:

$$\omega_{abc}^A = \frac{1}{3}(\omega_{abc} + \omega_{bca} + \omega_{cab}),\ \omega_{abc}^{S_1} = \frac{1}{3}(\omega_{abc} + \omega_{bac}),\ \omega_{abc}^{S_2} = \frac{1}{3}(\omega_{abc} + \omega_{cba}).$$

Defining for convenience

$$\bar{\omega}_b = 2\omega_{aba}, \quad \tilde{\omega}_a = \epsilon_{abcd}\omega_{bcd}^A,$$

one obtains after simple Clifford algebra:

$$\omega_{abc}^A\gamma_a\gamma_b\gamma_c = -\gamma_d\gamma_5\tilde{\omega}_d,\ \ \omega_{abc}^{S_1}\gamma_a\gamma_b\gamma_c = -\frac{1}{3}\bar{\omega}_c\gamma_c,\ \ \omega_{abc}^{S_2}\gamma_a\gamma_b\gamma_c = -\frac{2}{3}\bar{\omega}_b\gamma_b,$$

with $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$. Therefore, (1) and its conjugate become

$$\gamma_a\partial_a\Psi + \frac{1}{4}\gamma_a(\bar{\omega}_a + \gamma_5\tilde{\omega}_a)\Psi - iA_a\gamma_a\Psi = 0, \tag{3}$$

$$\partial_a\bar{\Psi}\gamma_a - \frac{1}{4}\bar{\Psi}(\bar{\omega}_a + \gamma_5\tilde{\omega}_a)\gamma_a + iA_a\bar{\Psi}\gamma_a = 0 \tag{4}$$

where $\bar{\Psi} = \Psi^\dagger\gamma_4$. Using (3) and (4), the $Spin^c(4)$ potential $\omega_\mu + iA_\mu$ may be decomposed as follows:

(A) $\bar{\Psi}\gamma_b\times$Eq.(3) $-$ Eq.(4)$\times\gamma_b\Psi$ leads to

$$\begin{aligned} 0 = {} & [(\bar{\Psi}\partial_b\Psi - \partial_b\bar{\Psi}\Psi) - 2\partial_a(\bar{\Psi}I_{ab}\Psi)] \\ & + \frac{1}{2}\bar{\omega}_b\rho\cosh\theta + \frac{1}{2}\tilde{\omega}_b\rho\sinh\theta - 2iA_b\rho\cosh\theta; \end{aligned} \tag{5}$$

(B) $\bar{\Psi}\gamma_b\gamma_5 \times$Eq.(3) $-$ Eq.(4)$\times\gamma_5\gamma_b\Psi$ leads to

$$\begin{aligned} 0 &= [(\bar{\Psi}\gamma_5\partial_b\Psi - \partial_b\bar{\Psi}\gamma_5\Psi) - 2\partial_a(\bar{\Psi}\gamma_5 I_{ab}\Psi)] \\ &\quad + \frac{1}{2}\bar{\omega}_b\rho\sinh\theta + \frac{1}{2}\tilde{\omega}_b\rho\cosh\theta - 2iA_b\rho\sinh\theta. \end{aligned} \tag{6}$$

Here we have introduced two real parameters ρ and θ,

$$\rho = \sqrt{(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2}, \quad (\cosh\theta = \frac{\bar{\Psi}\Psi}{\rho},\ \sinh\theta = \frac{\bar{\Psi}\gamma_5\Psi}{\rho}),$$

and applied the so-called duality rotation

$$e^{\gamma^5\theta} = \cosh\theta + \gamma^5\sinh\theta.$$

Eqs.(5) and (6) yield the following decomposition expression [7]:

$$\begin{aligned} & \omega_\mu + iA_\mu \\ = \; & -\frac{1}{\rho}e^{-\gamma_5\theta}e_{\mu b}[\frac{1}{2}(\partial_b\bar{\Psi}\Psi - \bar{\Psi}\partial_b\Psi) + \partial_a(\bar{\Psi}I_{ab}\Psi) \\ & + \frac{1}{2}\gamma_5(\partial_b\bar{\Psi}\gamma_5\Psi - \bar{\Psi}\gamma_5\partial_b\Psi) + \gamma_5\partial_a(\bar{\Psi}\gamma_5 I_{ab}\Psi)]. \end{aligned} \tag{7}$$

Following the routine of general relativity, a coordinate condition should be adopted for the vierbein $e_{\mu b}$ in (7) as a restraint. One chooses the harmonic coordinate condition [8]

$$\frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}g^{\nu\lambda}) = 0,$$

i.e.,

$$e^\mu_a\partial_\mu e^\nu_a + \frac{1}{2}\omega_a e^\nu_a = 0.$$

After some algebra, we obtain the desired decomposition of the $Spin^c(4)$ gauge potential

$$\begin{aligned} \omega_\mu + iA_\mu &= -\frac{\bar{\Psi}\Psi - \gamma_5\bar{\Psi}\gamma_5\Psi}{(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2}[j_\mu + \gamma_5\ \tilde{j}_\mu + \\ &\quad + \nabla_\nu(\bar{\Psi}\gamma^\nu{}_\mu\Psi) + \gamma_5\nabla_\nu(\bar{\Psi}\gamma^\nu{}_\mu\gamma_5\Psi)], \end{aligned} \tag{8}$$

where $\nabla_\mu = \partial_\mu + \Gamma_\mu$ is the affine covariant derivative with the Christoffel connection $\Gamma^\lambda_{\mu\nu}$ on $\mathcal{M}$ satisfying $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, and j_μ and $\tilde{j}_\mu$ are the quantum mechanical velocity-current and pseudo-current respectively given by

$$\begin{aligned} j_\mu &= \frac{1}{2}[(\partial_\mu\bar{\Psi})\Psi - \bar{\Psi}\partial_\mu\Psi], \\ \tilde{j}_\mu &= \frac{1}{2}[(\partial_\mu\bar{\Psi})\gamma_5\Psi - \bar{\Psi}\gamma_5\partial_\mu\Psi]. \end{aligned}$$

In the following sections we will study inner structure and topology of the $Spin^c(4)$ bundle by using the expression (8). Obviously, independent significance should be attached to the spinor structure of the $SO(4)$ potential when $A_\mu = 0$, since it may be applied to study many other physical and mathematical problems [4], which is beyond the scope of the present paper.

3 Determinant Equation for Twisting $U(1)$ Potential

The $Spin^c(4)$ space may be written as a direct sum $Spin^c(4) = (S^+ \otimes L) \oplus (S^- \otimes L)$, and the wave functions in $S^+ \otimes L$ and $S^- \otimes L$ are respectively the spinors $\Psi_\pm$: $\gamma_5 \Psi_\pm = \pm\Psi_\pm$. In the following we consider only the self-dual $S^+ \otimes L$. Thus for simplicity of notation, 'Ψ_+' will be denoted by 'Ψ'.

Let us study the $Spin^c(4)$ and its sub-bundle $SU(2)_+$. Noticing that γ_5 takes value $+1$ in this sub-space, we acquire from (3) and (4)

$$\omega_{+\mu} + iA_\mu = -\frac{2I}{\bar{\Psi}\Psi}[\frac{1}{2}[(\partial_\mu\bar{\Psi})\Psi - \bar{\Psi}\partial_\mu\Psi] + \nabla_\nu(\bar{\Psi}\gamma^\nu{}_\mu\Psi)]. \tag{9}$$

However, it is observed that the RHS of (9) is a $\frac{0}{0}$-indeterminate type since

$$\bar{\Psi}\Psi = (\partial_\mu\bar{\Psi})\Psi = \bar{\Psi}\partial_\mu\Psi = \bar{\Psi}\gamma^\nu{}_\mu\Psi = 0$$

[e.g., $\bar{\Psi}\Psi = -(\Psi^\dagger\gamma_5)\gamma_4(\gamma_5\Psi) = -\bar{\Psi}\Psi$]. In order to determine $\omega_{+\mu} + iA_\mu$, firstly we choose the realization for the γ-matrices:

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \; \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \; \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(with σ_i, $i = 1,2,3$, being the Pauli matrices), so that the self-dual Ψ is realized as $\Psi = (\ \Phi,\ \ \Phi\ )^T$ with $\bar{\Psi} = (\ \Phi^\dagger,\ \ -\Phi^\dagger\ )$, where Φ is a Pauli 2-spinor. Secondly, since the indeterminacy arises from the self-duality of Ψ, we introduce an anti-self-dual perturbation for Ψ:

$$\begin{pmatrix} \Phi \\ \Phi \end{pmatrix} \to \begin{pmatrix} (1+\varepsilon)\Phi \\ (1-\varepsilon)\Phi \end{pmatrix} \quad (\varepsilon \to 0^+).$$

Then, from (8), and defining

$$(e_k^\nu e_{\mu 4})_+ = \frac{1}{2}[e_k^\nu e_{\mu 4} + {}^*(e_k^\nu e_{\mu 4})] = \frac{1}{2}[e_k^\nu e_{\mu 4} + \frac{1}{2}\epsilon_{ijk4} e_i^\nu e_{\mu j}],$$

one determines

$$\omega_{+\mu} + iA_\mu = \frac{I}{2\Phi^\dagger\Phi}(\Phi^\dagger\partial_\mu\Phi - \partial_\mu\Phi^\dagger\Phi) - i\frac{2I}{\Phi^\dagger\Phi}\nabla_\nu[(e_k^\nu e_{\mu 4})_+\Phi^\dagger\sigma_k\Phi]. \tag{10}$$

Similarly, the second SW equation (2) may be regularized as

$$\partial_\mu A^\nu - \partial^\nu A_\mu = F_\mu{}^\nu = -(e_k^\nu e_{\mu 4})_+ \Phi^\dagger \sigma_k \Phi I. \tag{11}$$

Hence (10) becomes

$$\omega_{+\mu} + iA_\mu = \frac{I}{2\Phi^\dagger\Phi}(\Phi^\dagger \partial_\mu \Phi - \partial_\mu \Phi^\dagger \Phi) - i\frac{2}{\Phi^\dagger \Phi}\Delta A_\mu, \tag{12}$$

where $\Delta = \frac{1}{\sqrt{g}}\partial_\nu(g^{\nu\lambda}\sqrt{g}\partial_\lambda)$ is the Laplacian operator, and we have imposed the Lorentz gauge $\nabla_\nu A^\nu = 0$.

On the other hand, from the $SU(2)$ covariant derivative of Φ,

$$D\Phi = d\Phi - \varpi\Phi \quad (\varpi = \frac{1}{2i}\varpi_\mu^a \sigma_a dx^\mu),$$

the $SU(2)$ potential ϖ may be decomposed as [3]

$$\varpi_\mu = [\mathfrak{j}_\mu(\Phi) - \frac{1}{2}Tr(\mathfrak{j}_\mu(\Phi))I], \tag{13}$$

where

$$\mathfrak{j}_\mu(\Phi) = \frac{1}{\Phi^\dagger\Phi}(\partial_\mu \Phi \Phi^\dagger - \Phi \partial_\mu \Phi^\dagger)$$

and the parallel field condition $D_\mu\Phi = 0$ has been adopted.

Now, comparing the first term of the RHS of (12) with (13), it is recognized that they are identical in the sense that they lead to the same topological characteristic class (see the following Sect.4). Therefore from (12) we find an interesting determinant equation for the twisting potential A_μ:

$$\Delta A_\mu = -\lambda A_\mu \quad (\lambda \equiv \frac{1}{2}\Phi^\dagger\Phi). \tag{14}$$

Apparently, if (14) holds then (12) is an $SU(2)$ decomposition; namely, one realizes $Spin(4) = SU(2)_+ \otimes SU(2)_-$ by putting the restriction (14) on the twisting $U(1)$ potential A_μ. The appearance of Eq.(14) may be understood as follows. If the $Spin^c(4)$ connection ω_μ is regarded as a quantity determined by A_μ and Ψ via Eq.(2), then (2) will no longer serve as a constraint equation for the two unknowns A_μ and Ψ. Instead one has to pursue another constraint condition. (14) seems to be a reasonable choice.

Eq.(14) is an eigenvalue problem of the Laplacian Δ, with λ being the eigenvalue and A_μ the eigenfunction. This sort of problem has been investigated by many authors [9], where important theorems on the spectrum $\{\lambda_{(i)}, f_{(i)}; i = 0, 1, 2, \cdots\}$ (especially theorems on estimates and properties of the eigenvalues) have been obtained. Actually, if the base manifold has no boundary (which is the case for the SW theory), then Δ is an order-2 elliptic self-adjoint operator with discrete eigenvalues $\{0 = \lambda_{(0)} < \lambda_{(1)} \leq \lambda_{(2)} \leq \cdots\}$, while the non-trivial eigenfunctions $\{f_{(i)} \in C^\infty(\mathcal{M})\}$ form an orthogonal basis in the Hilbert space.

There are three points that should be addressed. Firstly, $\{\lambda_{(i)}\}$ is by definition the gauge-invariant vacuum expectation value of the self-dual wave function Ψ and therefore closely related to the crucial parameter $u = Tr\left\langle \phi^2 \right\rangle$ of the SW curves [5]. This establishes a direct relationship between the spectrum of Laplacian operator and the classification of the moduli space of the Dirac operator. Secondly, the sub-solution space corresponding to the zero-eigenvalue $\lambda_{(0)} = \frac{\Phi^\dagger\Phi}{2} = 0$ (i.e. $|\Phi| = 0$) is finite-dimensional; this case will be discussed in details in Sect.4 by showing that the zero-points of Φ correspond to instanton solutions with their topological charges accounting for the second Chern numbers (i.e. the Euler characteristics). Thirdly, the Laplacian Δ is an unbounded operator (since $\lambda_{(i)}\big|_{i\to\infty} \to \infty$), hence one introduces a heat operator $e^{-t\Delta}$ which is bounded since $e^{-t\lambda_{(i)}}\big|_{i\to\infty} \to 0$ for $\mathrm{Re}\, t > 0$. Defining $F_{(i)} = e^{-t\Delta} f_{(i)}$ from the above eigenfunctions $\{f_{(i)}\}$, it is easy to check that $F_{(i)}$ serves as the solution of the heat equation $(\Delta + \partial_t)F = 0$, where the solution F is related to the so-called heat kernel function $H(\vec{x}, \vec{y}, t)$ appearing in the expansion $(e^{-t\Delta} f)(x) = \int_{\mathcal{M}} H(\vec{x}, \vec{y}, t) f(\vec{y})$ $(\forall f \in L^2(\mathcal{M}))$, whose concrete form will depend on the background geometry of $\mathcal{M}$ [9].

4 Topological Characteristics of Instantons

When (14) holds, (12) gives the inner structure of the $SU(2)_+$ gauge potential

$$\omega_{+\mu} = \frac{I}{2\Phi^\dagger\Phi}(\Phi^\dagger\partial_\mu\Phi - \partial_\mu\Phi^\dagger\Phi).$$

In order to simplify this expression, one employes the component form of the Pauli spinor Φ,

$$\Phi = \begin{pmatrix} \phi^0 + i\phi^1 \\ \phi^2 + i\phi^3 \end{pmatrix},$$

with $\phi^A \in \mathbb{R}$ and $\Phi^\dagger\Phi = \phi^A\phi^A = \|\phi\|^2 \quad (A = 0, 1, 2, 3)$, and introduces a unit vector

$$n^A = \frac{\phi^A}{\|\phi\|} \quad (n^A n^A = 1).$$

Obviously the zero-points of ϕ^A are the singular points of n^A. Defining a normalized spinor

$$\hat{\Phi} = \frac{1}{\sqrt{\Phi^\dagger \Phi}} \Phi = \begin{pmatrix} n^0 + in^1 \\ n^2 + in^3 \end{pmatrix},$$

then one may write $\omega_{+\mu}$ as a 1-form

$$\omega_+ = \hat{\Phi}^\dagger d\hat{\Phi}.$$

It will be shown in the following that it is this simple expression of ω_+ that contributes to the second Chern classes and the Euler characteristics of the $SU(2)_+$ sub-bundle.

From the Chern-Simons 3-form constructed from ω_+

$$\Omega_+ = \frac{1}{8\pi^2} Tr(\omega_+ \wedge d\omega_+ - \frac{2}{3}\omega_+ \wedge \omega_+ \wedge \omega_+)$$

(see Ref.[3] and references therein), one acquires the spinor structure of the $SU(2)_+$ sub-bundle

$$\Omega_+ = \frac{1}{4\pi^2} \hat{\Phi}^\dagger d\hat{\Phi} \wedge d\hat{\Phi}^\dagger \wedge d\hat{\Phi}. \tag{15}$$

Furthermore, the second Chern class C_{2+} is given by

$$C_{2+} = d\Omega_+ = \rho_+(x) d^4x,$$

with $\rho_+(x)$ the so-called Chern density. Using (15) one gets

$$C_2 = \frac{1}{4\pi^2} d\hat{\Phi}^\dagger \wedge d\hat{\Phi} \wedge d\hat{\Phi}^\dagger \wedge d\hat{\Phi}, \tag{16}$$

i.e. the Chern density

$$\rho_+(x) = -\frac{1}{4\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \hat{\Phi}^\dagger \partial_\nu \hat{\Phi} \partial_\lambda \hat{\Phi}^\dagger \partial_\rho \hat{\Phi}.$$

This expression is consistent with the Pauli spinor structure of the $SU(2)$ Chern density obtained in Ref.[3]. Eqs.(15) and (16) also demonstrate from a different viewpoint that the above determinant equation (14) is a reasonable choice in the sense of the fractionalization $Spin^c(4) = (S^+ \otimes L) \oplus (S^- \otimes L)$.

In terms of the components of $\hat{\Phi}$,

$$\rho_+(x) = -\frac{1}{12\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ABCD} \partial_\mu n^A \partial_\nu n^B \partial_\lambda n^C \partial_\rho n^D,$$

which can be shown to be the form

$$\rho_+(x) = -\delta^4(\vec{\phi}) D(\frac{\phi}{x}), \tag{17}$$

where $D(\phi/x)$ is a Jacobian

$$\epsilon^{ABCD} D(\frac{\phi}{x}) = \epsilon^{\mu\nu\lambda\rho} \partial_\mu \phi^A \partial_\nu \phi^B \partial_\lambda \phi^C \partial_\rho \phi^D.$$

Eq.(17) implies that

$$\rho_+(x) \begin{cases} = 0, & \text{iff } \vec{\phi} \neq 0, \\ \neq 0, & \text{iff } \vec{\phi} = 0. \end{cases}$$

So it is sufficient to study the zero-points of $\vec{\phi}$ to determine the non-zero solutions of $\rho_+(x)$. Meanwhile, we should bear in mind that $\|\phi\| = 0$ corresponds to the zero-eigenvalue of Eq.(14).

The implicit function theory shows [10] that under the regular condition $D(\phi/x) \neq 0$, the general solutions of

$$\phi^A(x^1, x^2, x^3, x^4) = 0 \quad (A = 0, 1, 2, 3)$$

are expressed as N isolated points

$$x^\mu = x_j^\mu \qquad (j = 1, ..., N),$$

around which $\delta^4(\vec{\phi})$ is expanded as [11]:

$$\delta^4(\vec{\phi}) = \sum_{j=1}^{N} \left. \frac{\beta_j \delta^4(x^\mu - x_j^\mu)}{|D(\phi/x)|} \right|_{x_j^\mu}.$$

Here the positive integer β_j is the so-called Hopf mapping index, which means topologically that when a point x^μ covers the neighborhood of the jth zero-point x_j^μ once, the vector field ϕ^A will cover the corresponding region in the ϕ-space for β_j times. Then, introducing the Brouwer mapping degree

$$\eta_j = \left. \frac{D(\phi/x)}{|D(\phi/x)|} \right|_{x_j^\mu} = sign[D(\frac{\phi}{x})]_{x_j^\mu},$$

$\rho_+(x)$ is re-expressed as

$$\rho_+(x) = -\sum_{j=1}^{N} \beta_j \eta_j \delta^4(x^\mu - x_j^\mu),$$

which shows that the Chern density is non-zero only at the N 4-dimensional zero-points of ϕ^A (i.e. the singular points of n^A). These self-dual singular-points are regarded as instanton solutions in the $SU(2)_+$ sub-space, with their topological charges characterized by the Hopf index β_j and Brouwer degree η_j.

Integrating the second Chern class yields the second Chern number:

$$c_{2+} = \int C_{2+} = \int \rho_+(x) d^4x = -\sum_{j=1}^{N} \beta_j \eta_j.$$

Since the base manifold $\mathcal{M}$ is 4-dimensional, the second Chern class C_{2+} is simultaneously the top Chern class and the Euler class on $\mathcal{M}$. Hence the Euler characteristic, which is the sum of indices of zero-points of vector field ϕ^A on $\mathcal{M}$ (Poincaré-Hopf theorem), is obtained by the Gauss-Bonnet theorem:

$$\chi_+(\mathcal{M}) = \int C_{2+} = -\sum_{j=1}^{N} \beta_j \eta_j,$$

which demonstrates that the topological invariant $\chi_+(\mathcal{M})$ arises from the contribution of the topological charges of instantons.

5 Conclusion

In this paper we obtain the decomposition Eq.(8) for the $Spin^c(4)$ gauge potential $\omega_\mu + iA_\mu$. Comparing it with the decomposition of the $SU(2)$ potential ϖ_μ, we find a determinant equation (14) for the twisting $U(1)$ potential A_μ, which is in itself an eigenvalue equation of the Laplacian operator, with the eigenvalue being the vacuum expectation value of Ψ. This establishes a relationship between the spectral theory of Laplacian operator and the classification of the moduli space of the Dirac operator appearing in the SW equations. Moreover, topological charges of instantons expressed with the characteristic numbers β_j and η_j are discussed.

Some remarks are in order. In Sect.3 we extracted (14) from (12). Another interesting choice for the $U(1)$ potential in (12) is $A_\mu = \bar{A}_\mu + \alpha_\mu$, where $\bar{A}_\mu$ satisfies (14) and α_μ leads to a δ-function like field strength: $d\alpha \propto \delta^2(\vec{\varphi})$, with $\vec{\varphi}$ being a vector defined from the spinor Φ in the $SU(2)_+$ sub-space. A choice for α_μ is the Wu-Yang potential [4]: $\alpha_\mu \propto \vec{e}_1 \cdot \partial_\mu \vec{e}_2$, where $(\vec{e}_1, \vec{e}_2, \vec{m})$ is an orthogonal frame with $\vec{m} = \frac{\Phi^\dagger \vec{\sigma} \Phi}{\Phi^\dagger \Phi}$ defined as a 3-unit vector in the $SU(2)_+$ sub-space. Then, α_μ will contribute to the LHS of (12) a Chern-class term containing a 2-dimensional δ-function which indicates the impact of the 2-dimensional zero-points of $(\vec{e}_1, \vec{e}_2)$, and contribute to the RHS a Schwinger term related to a central extension which describes the 3-dimensional zero-points of $\vec{m}$.

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References

[1] L. Faddeev and A. J. Niemi, *Partially dual variables in $SU(2)$ Yang-Mills theory*, *Phys. Rev. Lett.* **82** (1999) 1624;
ibid., *Decomposing the Yang-Mills field*, *Phys. Lett.* **B 464** (1999) 90;
ibid., *Aspects of electric and magnetic variables in $SU(2)$ Yang-Mills theory*, *Phys. Lett.* **B 525** (2002) 195;
Y.M. Cho, *Magnetic confinement in QCD, J. Korean Phys. Soc.* **38** (2001) 151.

[2] Y.S. Duan, X. Liu and P.M. Zhang, *Decomposition theory of the $U(1)$ gauge potential and the London assumption in topological quantum mechanics, J. Phys. Condensed Matter* **14** (2002) 7941.

[3] Y.S. Duan and X. Liu, *Spinor Decomposition of SU(2) Gauge Potential and Spinor Structures of Chern Simons and Chern Density, Commun. Theor. Phys.* **40** (2003) 447 [math-ph/0201018].

[4] Y.S. Duan and X. Liu, *Knotlike cosmic strings in the early universe, JHEP* **02** (2004) 028;
X. Liu, Y.Z. Zhang, Y.S. Duan and L.B. Fu, *Various topological excitations in the $SO(4)$ gauge field in higher dimensions, Ann. Phys.* **318** (2005) 419.

[5] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang-Mills theory, Nucl. Phys.* **B 426** (1994) 19;
ibid., *Monopoles, duality and chiral symmetry breaking in $\mathcal{N}=2$ supersymmetric QCD, Nucl. Phys.* **B 431** (1994) 484;
L. Alvarez-Gaumé and S.F. Hassan, *Introduction to S-duality in $\mathcal{N}=2$ supersymmetric gauge theory. (A pedagogical review of the work of Seiberg and Witten)*, hep-th/9701069;
G. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. **1** (1997) 298 [hep-th/9709193].

[6] E. Witten, *Monopoles and Four-Manifolds, Math. Research Letters* **1** (1994) 769;
C.H. Taubes, *The Seiberg-Witten invariants and symplectic forms, Math. Research Letters* **1** (1994) 809;
J.W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds,* Princeton University Press, Princeton, 1996;
G.P. Collins, *Supersymmetric QCD sheds light on quark confinement and the topology of* 4*-manifolds, Phys. Today* **48** (1995) 17.

[7] A.A. Campolattaro, *New Spinor Representation of Maxwell's Equations. I. Generalities, Int. J. Theor. Phys.* **19** (1980) 99;
ibid., New Spinor Representation of Maxwll's Equations. II. Algebraic Properties, Int. J. Theor. Phys. **19** (1980) 127;
ibid., Generalized Maxwell Equations and Quantum Mechanics. I. Dirac Equation for the Free Electron, Int. J. Theor. Phys. **29** (1990) 141;
ibid., Generalized Maxwell's Equations and Quantum Mechanics. II. Generalized Dirac Equation, Int. J. Theor. Phys. **29** (1990) 147.

[8] W. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[9] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry,* International Press, Cambridge MA, 1994;
T. Aubin, *Some Nonlinear Problems in Riemannian Geometry,* Springer, Berlin, 1998;
P.B. Gilkey, *Invariance Theory, The Heat Equation, and The Atiyah-Singer Index Theorem,* Publish and Perish, USA, 1984.

[10] S.G. Krantz, H.R. Parks, *The Implicit Function Theorem,* Birkhäuser, Boston, 2003.

[11] J.A. Schouten, *Tensor Analysis for Physicists,* Clarendon, Oxford, 1951.